Two New Approximate Methods for Analyzing Free Vibration of Structural Components

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Two approximate methods, which have not previously been used for structural dynamics problems, are applied to the free vibration analysis of various structural components. The first method is a new version of the complementary energy method. It is shown to be considerably more accurate than the conventional Rayleigh and Rayleigh-Schmidt methods when applied to spatially one-dimensional free vibration problems: prismatic and tapered bars, prismatic beams, and axisymmetric motion of circular membranes. The second method is the differential quadrature method introduced by Bellman and his associates. It is applied successfully here to all of the problems mentioned plus square membranes and circular and square plates.

Nomenclature

$A(X)$ = cross-sectional area
A_{ij} = weighting coefficient
A_0 = constant value of A
a = edge radius of circle; half-length of rectangle
B_{ij} = weighting coefficient
b = half-width of rectangle
C, C_1, C_2 = constants of integration
C_{ij} = weighting coefficient
D = flexural rigidity of plate
D_{ij} = weighting coefficient
E = Young's modulus
e = base of the natural logarithms
f = function
h = thickness of membrane or plate
I_0 = area moment of inertia about neutral axis (constant)
i = $\sqrt{-1}$
i, j, k = indices
L = linear differential operator
ℓ = length of bar or beam
$M(X)$ = bending moment amplitude
$m(X)$ = mass per unit length
m_0 = constant value of m
N = upper limit of summation
N_0 = tension (force/width)
n = exponent
$P(X)$ = axial force amplitude
p = reversed effective distributed pressure amplitude (normal force/area)
q = reversed effective distributed loading amplitude (normal force/length)
R = dimensionless radial position ($= r/a$)
r = radial position
T_{max} = maximum kinetic energy
t = time
U_{max} = maximum complementary energy
u = axial displacement
$V(X)$ = transverse shear force amplitude
W = modal displacement
W_i = value of W at a grid point
w = normal deflection
X = dimensionless axial position ($= x/\ell$ or x/a)
x = axial position (measured from end of bar or beam, or from center of rectangle)
Y = dimensionless transverse position ($= y/b$)
y = transverse position in plane of rectangle (measured from center)
α = aspect ratio of rectangle ($= a/b$)
ρ = material density
ω = circular natural frequency
$\bar{\omega}$ = dimensionless value of ω
$(\ )'$ = $d(\ )/dX$

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Introduction

TODAY'S structural dynamics specialists have a wide variety of approximate methods of analysis available to them.[1] These include the traditional Galerkin and Rayleigh-Ritz methods and the more "numerical" methods such as finite differences, transfer matrices, finite elements, and, more recently, boundary elements. The traditional methods, although less expensive computationally, lack accuracy, while the numerical methods when applied in fine mesh sizes are accurate but computationally expensive. With the advent of inexpensive but powerful small computers, a new avenue of analysis looks promising: it seeks a happy middle ground between the traditional approximate methods and the numerical ones.

Two methods in this new category are investigated in this context here. One is an original method representing an energy technique that combines ideas previously used individually by Rayleigh[2] and Schmidt[3] and by Bhat.[4] The other is the differential quadrature method introduced by Bellman and Casti[5] in 1971. To the best of the present investigators' knowledge, this latter method has not been applied previously to structures problems or to any problems involving partial differential equations (PDE) higher than second order.

A New Version of the Energy Method

The Rayleigh and Rayleigh-Ritz energy methods are widely used for obtaining approximate values for the natural frequencies and mode shapes of elastic structures. Although specific mode shape terms are generally applied in these methods, this is not essential. In fact, as was shown by Rayleigh[2] and Schmidt,[3] power functions with nonspecified exponents may be used and the exponents optimized to give a minimum value of the natural frequency, since the energy methods result in upper bounds for frequency. In a recent survey of the application of this Rayleigh-Schmidt technique, Bert[6] referenced approximately 30 papers in which the technique was used for vibration problems of beams, membranes, and plates.

The idea of using complementary energy instead of potential (strain) energy in vibration problems has a long history going back to the work of Grammel[7] in 1939. The work of Westergaard,[8] Reissner,[9] and Niedenfuhr[10] should also be mentioned. Additional references are listed in Tabarrok's recent survey paper.[11]

It was only relatively recently that Kirkhope[12] for shear-deformable rings and Chang[13] for axial vibration of bars introduced a new, very simple method based on an equivalent distributed inertial force (d'Alembert reversed effective force) determined from the assumed displacement function. Then, from quasistatic equilibrium considerations, the generalized forces are found and used to express the energy. This approach was further refined and simplified by Bhat.[4]

The present method can be considered to be a synthesis of the Rayleigh-Schmidt and Bhat ideas. By application to three diverse examples, it is shown that this method can be quite accurate (fundamental frequency within less than 0.05% of the exact value with a single-term mode shape).

First, using the Rayleigh-Schmidt idea, a single-term modal displacement function with an undetermined power is assumed. Next, the d'Alembert reversed effective force idea is used to express the reversed effective force corresponding to the trial displacement. Subsequently, the maximum complementary energy is established. In the case of axial vibration of a bar, this may be done directly. However, in the case of axisymmetric vibration of a membrane, it is necessary to integrate an equilibrium equation and invoke a regularity condition. For a beam, to obtain the complementary energy expression, it is necessary to integrate two equilibrium equations and invoke two boundary conditions. Then, the maximum kinetic energy is expressed in terms of the displacement function. Finally, equating the maximum complementary and kinetic energies yields an expression for the natural frequency corresponding to the assumed modal displacement. The optimum value of the undetermined exponent is the one corresponding to the lowest value of the modal frequency.

It is noted that the assumed modal displacement function must satisfy geometric or kinematic boundary conditions, but not necessarily the natural or kinetic boundary conditions. However, additional satisfaction of the latter should be expected to yield more accurate results. Also, it should be emphasized that this method, like other energy methods, is an upper bound method, i.e., it predicts a frequency that approaches the exact frequency from above.

Differential Quadrature Method

Any solutions obtained by the standard finite-difference method, finite-element method, or boundary-element method usually have to be computed based on a large number of surrounding points, since the accuracy and stability of the mentioned methods depend strongly on the nature and refinement of the discretization of the domain. Consequently, computational efforts are often prohibitive for these numerical techniques.

In order to circumvent the above difficulties, an efficient procedure called the differential quadrature method (DQM) was introduced by Bellman et al.[5,14–18] for the rapid solution of linear and nonlinear partial differential equations. Mingle[19,20] applied the DQM to the solutions of the nonlinear diffusion equation. Civan and Sliepcevich[21–23] extended and generalized the method of differential quadrature and applied it to transport processes, the Poisson equation, and multidimensional problems.

In the present paper, the DQM is applied to various bar, beam, membrane, and plate vibration problems and the results obtained are compared with exact solutions, where available, and with approximate solutions obtained by other methods.

The DQM approximates the partial derivative of a function at a given discrete point as a weighted linear sum of function values at all given discrete points. Thus, for time-independent problems and linear free vibration problems, the partial differential equation is reduced to a set of algebraic equations. This method, as any polynomial approach, is dependent on the order of the approximating polynomial, i.e., with an increase in the order of the polynomial, the accuracy of the solution increases. Possible oscillations arising from higher-order polynomials can be avoided by using numerical interpolation methods. A differential quadrature approximation at the ith discrete point is given by

$$L\{f(x)\}_i \cong \sum_{j=1}^{N} A_{ij} f(x_j) \tag{1}$$

in which L is a linear differential operator applied to a function, $f(x)$, x_i are the discrete points in the variable domain, $f(x_j)$ the function values at these points, and A_{ij} the weighting coefficients attached to these function values.

In order to determine the weighting coefficients A_{ij}, it is required that Eq. (1) be exact for all polynomials of degree less than or equal to $(N-1)$. The general form of a test function is then given by

$$f_k(x) = x^{k-1}, \qquad k = 1,2,\ldots,N \tag{2}$$

so that Eq. (1) leads to

$$L\{x^{k-1}\}_i \cong \sum_{j=1}^{N} A_{ij} x_j^{k-1}, \qquad i,k = 1,2,\ldots,N \tag{3}$$

If the linear differential operator L represents an nth derivative, then the left side of Eq. (3) can be replaced as follows:

$$L\{x^{k-1}\}_i = (k-1)(k-2)\ldots(k-n)x_i^{(k-n-1)} \qquad i,k = 1,2,\ldots,N \tag{4}$$

Substituting Eq. (4) into Eq. (3) yields

$$\sum_{j=1}^{N} A_{ij} x_j^{k-1} = (k-1)(k-2)\ldots(k-n)x_i^{(k-n-1)} \qquad i,k = 1,2,\ldots,N \tag{5}$$

This expression represents N sets of N linear algebraic equations for the determination of the weighting coefficients A_{ij}. It is noted that the sets have a unique solution for the weighting coefficients A_{ij}, since the matrix of elements x_j^{k-1} represents a Vandermonde matrix which always has an inverse as described by Hamming.[24] The weighting coefficients A_{ij} are then used in Eq. (1) to express the derivatives of a function at a discrete point in terms of all the discrete function values. It is emphasized that the quadrature must be of higher order than the order of any partial derivative, i.e., $N > n$.

Multidimensional problems in more than one space variable can be treated in essentially the same way by using linear transformations with respect to the space variables for the partial derivatives. For higher-order partial derivatives, the computational effort required for the determination of the weighting coefficients increases somewhat, since the approximation formulas have to be obtained by iteration of these transformations. Thus, for this case, a method originated by Mingle[19] is suggested; it essentially utilizes separate transformations for derivatives of higher order. It is noted that since the boundary conditions are applied both at the boundaries and also at very small distances δ from the boundaries, an attempt to make a polynomial approach points that are not on the polynomial will cause the polynomial to oscillate. These oscillations get larger as the degree of the polynomial is increased. Thus, the cure for a poor polynomial fit seems to be to stay with lower order polynomials. Finally, for application to linear structural vibration problems, time is easily eliminated by the use of normal modes.

Applications to Structural Vibration Problems

To illustrate the computational simplicity and efficiency of the two methods previously described, they are applied to a series of linear structural vibration problems.

Example 1: Axial Vibration of a Slender Bar

The bar is considered to be free at the coordinate origin ($x=0$) and fixed at the other end ($x=\ell$). The axial displacement can be expressed as

$$u(x,t) = W(X)\,e^{i\omega t} \tag{6}$$

where t is the time, W the modal displacement, X the dimensionless position coordinate ($=x/\ell$), and ω the natural frequency.

In the new energy method (NEM), the modal displacement function is selected to be

$$W(X) = X^n - 1 \tag{7}$$

which satisfies both of the boundary conditions

$$W'(0) = 0,\; W(1) = 0 \tag{8}$$

where ( )$'$ denotes d( )/dX.

Using the d'Alembert reversed effective force idea, the axial force at dimensionless position X is

$$P(X) = -\ell\int_0^X m(X)\,\omega^2 W(X)\,\mathrm{d}X + C_1 \tag{9}$$

where $m(X)$ is the mass per unit length.

Since the bar is free at $X=0$, the constant of integration $C_1=0$.

The maximum complementary and kinetic energies are given by

$$U_{\max} = \left(\frac{\ell}{2}\right)\int_0^1 \frac{[P(X)]^2}{EA(X)}\,\mathrm{d}X \tag{10}$$

$$T_{\max} = \left(\frac{\ell}{2}\right)\int_0^1 m(X)\,\omega^2[W(X)]^2\,\mathrm{d}X \tag{11}$$

Substituting Eqs. (7) and (9) into Eqs. (10) and (11), performing the integrations, and equating $U_{\max}$ to $T_{\max}$ yield an expression for the circular natural frequency ω.

For the case of a prismatic bar, $m(X) = m_0 =$ const and $A(X) = A_0 =$ const. Then,

$$P(X) = -m_0\omega^2\ell\left[X^{n+1}/(n+1) - X\right]$$

$$U_{\max} = \frac{m_0^2\omega^4\ell^3}{2EA_0}\left[\frac{1}{(n+1)^2(2n+3)} - \frac{2}{(n+1)(n+3)} + \frac{1}{3}\right]$$

$$T_{\max} = \frac{m_0\omega^2\ell}{2}\left(\frac{1}{2n+1} - \frac{2}{n+1} + 1\right)$$

Thus,

$$\omega^2 = \left(\frac{EA_0}{m_0\ell^2}\right)\frac{[1/(2n+1)] - [2/(n+1)] + 1}{\dfrac{1}{(n+1)^2(2n+3)} - \dfrac{2}{(n+1)(n+3)} + \dfrac{1}{3}}$$

It is convenient to introduce the dimensionless frequency $\bar{\omega}$ as follows, so that it is a function only of the exponent n:

$$\bar{\omega}^2 = \left(m_0\ell^2/EA_0\right)\omega^2 \tag{12}$$

Then,

$$\bar{\omega}^2 = \frac{[(1/2n+1)] - [2/(n+1)] + 1}{\dfrac{1}{(n+1)^2(2n+3)} - \dfrac{2}{(n+1)(n+3)} + \dfrac{1}{3}} \tag{13}$$

If one selects various integer values for n, one finds that the best is $n=2$, which gives $\bar{\omega} = 1.5718$, only 0.064% higher than the exact value of $\pi/2$ (≈ 1.5708). However, the optimal value of n, which makes $\bar{\omega}$ a minimum, is $n=1.7$, which gives $\bar{\omega}=1.5709$, only 0.006% higher than the exact value.

To apply the DQM to axial vibration of a slender bar, one first writes the governing differential equation

$$E[A(X)W'(X)]' = -m(X)\,\omega^2 W(X) \tag{14}$$

and uses the boundary conditions expressed by Eqs. (8).

For a prismatic bar, Eq. (14) becomes

$$W''(X) = -\bar{\omega}^2 W(X) \tag{15}$$

In terms of differential quadrature, one obtains

$$W_N = 0 \tag{16a}$$

$$\sum_{j=1}^{N-1} B_{ij}W_j = -\bar{\omega}^2 W_i, \qquad i = 2,3,\ldots,(N-1) \tag{16b}$$

$$\sum_{j=1}^{N-1} A_{1j}W_j = 0 \tag{16c}$$

where B_{ij} and A_{ij} are the weighting coefficients for the second- and first-order derivatives, respectively. From this set of eigenvalue equations, the fundamental frequency can be easily evaluated. The use of eight grid points along the length of the bar yields $\bar{\omega} = 1.5709$, the same result previously obtained by the NEM.

As a second case of axial vibration of a bar, the tapered conical bar that Meirovitch[25] investigated by various methods is considered. The beam is tapered from a point at its free end such that both its area $A(X)$ and mass per unit length $m(X)$ vary linearly with dimensionless position X. Thus, $A(X) = 2A_0X$ and $m(X) = 2m_0X$. The boundary conditions are expressed by Eqs. (8).

For the new energy method, use of Eqs. (7), (9), (10), (11), and the preceding expressions for $A(X)$ and $m(X)$ and equating $U_{\max}$ and $T_{\max}$ yield the following expression for dimensionless fundamental frequency squared as a function of exponent n:

$$\bar{\omega}^2 \equiv \frac{\omega^2 m_0\ell^2}{EA_0} = \frac{8(n+2)^2(n+4)}{(n+1)(n^2+10n+20)} \tag{17}$$

For the differential quadrature method in conjunction with the varying cross-sectional area and unit mass and the boundary conditions, one obtains instead of Eqs. (16)

$$W_N = 0 \tag{18a}$$

$$\sum_{j=1}^{N-1} B_{ij}W_j + \frac{1}{X_i}\sum_{j=1}^{N-1} A_{ij}W_j = -\bar{\omega}^2 W_i, \qquad i = 2,3,\ldots,(N-1) \tag{18b}$$

$$\sum_{j=1}^{N-1} A_{1j} W_j = 0 \tag{18c}$$

Comparison of NEM and DQM results with those obtained by numerous other methods is presented in Table 1. The clear advantages of the present methods are demonstrated.

Example 2: Flexural Vibration of a Prismatic Beam

The slender beam is considered to be cantilevered, with the fixed end being the coordinate origin. The deflection, i.e., the transverse displacement, can be expressed as

$$w(x,t) = W(X)e^{i\omega t} \tag{19}$$

The geometric boundary conditions at the clamped end are

$$W(0) = 0, \qquad W'(0) = 0 \tag{20}$$

and the natural boundary conditions at the free end (no bending moment and no shear force) are

$$M(1) = 0 \Rightarrow W''(1) = 0$$

$$V(1) = 0 \Rightarrow W'''(1) = 0 \tag{21}$$

Using the NEM, the modal displacement function is taken to be

$$W(X) = X^n \tag{22}$$

which satisfies the two geometric boundary conditions (provided that $n > 1$) at $X = 0$, but not the two natural ones at $X = 1$.

The amplitude of the d'Alembert distributed loading is

$$q(X) = -\rho A_0 \omega^2 W(X) \tag{23}$$

where A_0 is the constant cross-sectional area and ρ the density. The transverse shear force amplitude is

$$V(X) = -\ell \int_0^X q(X)\,\mathrm{d}X = \frac{\rho A_0 \omega^2 \ell X^{n+1}}{n+1} + C_1 \tag{24}$$

Application of the boundary condition $V(1) = 0$ gives $C_1 = -\rho A_0 \omega^2 \ell/(n+1)$.

The bending moment amplitude is

$$M(X) = \ell \int_0^X V(X)\,\mathrm{d}X = \frac{\rho A_0 \omega^2 \ell^2 X^{n+2}}{(n+1)(n+2)} + C_1 \ell X + C_2 \tag{25}$$

Use of the boundary condition $M(1) = 0$ results in

$$C_2 = -C_1\ell - \rho A_0 \omega^2 \ell^2/[(n+1)(n+2)]$$

The maximum complementary energy is

$$U_{\max} = \ell \int_0^1 \frac{M^2\,\mathrm{d}X}{2EI_0} \tag{26}$$

where I_0 is the constant area moment of inertia about the neutral axis. The maximum kinetic energy is

$$T_{\max} = \frac{\ell}{2}\int_0^1 \rho A_0 \omega^2 W^2\,\mathrm{d}X \tag{27}$$

Thus, equating the complementary and kinetic energies after integration yields the following expression for the square of the dimensionless natural frequency:

$$\bar{\omega}^2 \equiv \frac{\rho A_0 \ell^4 \omega^2}{EI_0}$$

$$= \frac{(n+1)^2/(2n+1)}{\dfrac{1/(2n+5) + (n+1)/(n+3)}{(n+2)^2} - \dfrac{(n+1)^2/(2n+1)}{\dfrac{2}{n+2}\left(\dfrac{1}{n+4} - \dfrac{1}{n+3} + \dfrac{1}{2}\right) + \dfrac{1}{3}}} \tag{28}$$

It is obvious that $\bar{\omega}^2$ is a function of the exponent n only. Optimization of the value of n to obtain a minimum value yields the best prediction of $\bar{\omega}$.

To apply the DQM to flexural vibration of a slender prismatic beam, one writes the governing differential equation,

$$W^{iv}(X) = \bar{\omega}^2 W(X) \tag{29}$$

Thus, in differential quadrature terms,

$$\sum_{j=1}^{N} D_{ij} W_j = \bar{\omega}^2 W_i, \qquad i = 3, \ldots, (N-2) \tag{30}$$

With appropriate boundary conditions [Eqs. (20) and (21)], the dimensionless frequency $\bar{\omega}$ can be evaluated easily.

Table 2 summarizes the results obtained by the present NEM and DQM and a variety of other methods. Again, NEM and DQM are among the most accurate of the approximate methods.

Example 3: Transverse Vibrations of Taut Membranes

First, axisymmetric transverse vibrations of a solid circular membrane are considered. The membrane is fixed at its edge, has thickness h, and N_0 is the uniform tension per unit length. The radial position coordinate is denoted by r. The transverse displacement can be expressed as

$$w(r,t) = W(R)e^{i\omega t} \tag{31}$$

Table 1 Summary of numerical results for fundamental frequency of axial vibration of a free-fixed conical bar (exact result: $\bar{\omega} = 2.4048$[a])

Method	$\bar{\omega}$	% Error
Rayleigh		
Sinusoidal[b]	2.4146	0.41
Power ($n = 2$)	2.4495	1.86
Collocation[c]	2.3939	−0.45
Rayleigh-Schmidt ($n = 1.414$)	2.4142	0.39
Bhat ($n = 2$)[d]	2.4121	0.30
Present		
New energy method ($n = 1.38$)	2.4055	0.029
Differential quadrature (8 grid points)	2.4048	0

[a]Ref. 25, p. 211. [b]Ref. 25, p. 210. [c]Ref. 25, p. 241. [d]Ref. 4.

Table 2 Summary of numerical results for fundamental frequency of a prismatic cantilever beam (exact result: $\bar{\omega} = 3.5156$)

Method	$\bar{\omega}$	% Error
Rayleigh ($n = 2$)	4.472	27.2
Rayleigh-Schmidt ($n = 1.73$)[a]	3.932	11.8
Rayleigh-Schmidt FEM ($n = 2.8$)[b]	3.5173	0.048
Bhat[c]	3.5300	0.41
Present		
New energy method ($n = 1.6$)	3.5167	0.031
Differential quadrature (8 grid points)	3.5226	0.20

[a]Ref. 3. [b]Ref. 26. [c]Ref. 4.

where R is the dimensionless radius ($= r/a$), a being the edge radius.

The geometric boundary condition at the edge is

$$W(1) = 0 \tag{32}$$

However, since the membrane is solid, the following regularity condition must also be satisfied at the origin:

$$\mathrm{d}W(0)/\mathrm{d}R = 0 \tag{33}$$

Using the NEM, the modal displacement function is taken to be

$$W(R) = 1 - R^n \tag{34}$$

which satisfies Eqs. (32) and (33).

The amplitude of the reversed effective pressure is

$$p(R) = -\rho h \omega^2 W(R) \tag{35}$$

The equilibrium equation is

$$\left(\frac{N_0}{r}\right)\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}W}{\mathrm{d}r}\right) = -\rho h \omega^2 W \tag{36}$$

Substituting Eq. (34) into Eq. (36) and integrating, one obtains

$$\frac{\mathrm{d}W}{\mathrm{d}r} = -\left(\frac{\rho h \omega^2 a^2}{N_0}\right)\left(\frac{R}{2} - \frac{R^{n+1}}{n+2}\right) + \left(\frac{C}{R}\right) \tag{37}$$

where C is a constant of integration. However, to satisfy the regularity condition $\mathrm{d}W(0)/\mathrm{d}R = 0$, C must be set equal to zero.

The maximum complementary energy is

$$U_{\max} = \left(\frac{1}{2N_0}\right)\int_0^a \left(N_0 \frac{\mathrm{d}W}{\mathrm{d}r}\right)^2 (2\pi r)\,\mathrm{d}r \tag{38a}$$

or

$$U_{\max} = \left(\frac{\pi}{N_0}\right)\int_0^1 \left(N_0 \frac{\mathrm{d}W}{\mathrm{d}R}\right)^2 R\,\mathrm{d}R \tag{38b}$$

The maximum kinetic energy is

$$T_{\max} = \frac{1}{2}\rho h \omega^2 a^2 \int_0^1 W^2 (2\pi R)\,\mathrm{d}R \tag{39}$$

Using Eqs. (34) and (37) in Eqs. (38) and (39), performing the necessary integrations, and equating $U_{\max}$ to $T_{\max}$, one obtains

$$\bar{\omega}^2 \equiv \frac{\rho h a^2 \omega^2}{N_0} = \frac{\frac{1}{2} - \frac{2}{n+2} + \frac{1}{2n+2}}{\frac{1}{16} - \frac{1}{(n+2)(n+4)} + \frac{1}{2(n+2)^3}} \tag{40}$$

The minimum value of $\bar{\omega}$ is 2.405, which occurs at a value of 1.37 for n.

To apply the DQM, the governing differential equation for the modal displacement is written as

$$\frac{\mathrm{d}^2 W}{\mathrm{d}R^2} + \frac{1}{R}\frac{\mathrm{d}W}{\mathrm{d}R} + \bar{\omega}^2 W = 0 \tag{41}$$

Applying differential quadrature, one obtains

$$W_N = 0 \tag{42a}$$

$$\sum_{j=1}^{N-1} B_{ij} W_j + \frac{1}{R_i}\sum_{j=1}^{N-1} A_{ij} W_j = -\bar{\omega}^2 W_i, \qquad i = 2,\ldots,(N-1) \tag{42b}$$

$$\sum_{j=1}^{N-1} A_{1j} W_j = 0 \tag{42c}$$

Table 3 presents values of $\bar{\omega}$ as obtained by various methods.

For a rectangular membrane,

$$w(x,y,t) = W(X,Y)e^{i\omega t} \tag{43}$$

where $X = x/a$, $Y = y/b$, and the sides are $2a$ and $2b$. The governing differential equation for the modal displacement is

$$\frac{\partial^2 W}{\partial X^2} + \alpha^2 \frac{\partial^2 W}{\partial Y^2} = -\frac{1}{4}\bar{\omega}^2 W \tag{44}$$

where $\alpha = a/b$ is the aspect ratio and

$$\bar{\omega}^2 = \rho h (2a)^2 \omega^2 / N_0$$

Thus, the differential quadrature equations for a square membrane ($\alpha = 1$) are

$$\sum_{k=2}^{N-1} B_{ik} W_{kj} + \sum_{k=2}^{N-1} B_{jk} W_{ik} = -\frac{1}{4}\bar{\omega}^2 W_{ij}$$
$$i, j = 2,\ldots,(N-1) \tag{45}$$

in addition to the boundary condition equations.

The results for $\bar{\omega}$ for a square membrane as obtained by several methods are presented in Table 4.

Example 4: Transverse Vibrations of Thin, Isotropic Plates

The governing differential equation for this class of problems is

$$D\nabla^4 w = -\rho h \frac{\partial^2 w}{\partial t^2} \tag{46}$$

where D is the flexural rigidity and ∇^4 the biharmonic operator.

Table 3 Summary of numerical results for fundamental frequency of transverse vibration of a circular membrane (exact result: $\bar{\omega} = 2.404$)

Method	$\bar{\omega}$	% Error
Rayleigh ($n = 2$)[a]	2.449	1.87
Bhat ($n = 2$)[a]	2.412	0.33
Present		
New energy method ($n = 1.37$)	2.405	0.04
Differential quadrature (8 grid points)	2.405	0.04

[a] Ref. 4.

Table 4 Summary of numerical results for fundamental frequency of transverse vibration of a square membrane (exact result: $\bar{\omega} = \pi\sqrt{2} \approx 4.443$)

Method	$\bar{\omega}$	% Error
Rayleigh ($n = 2$)	4.472	0.65
Rayleigh-Schmidt ($n = 1.7$)	4.450	0.16
Present: Differential quadrature (5×5 grid points)	4.465	0.50

For the case of axisymmetric vibration of a circular plate, Eq. (31) is again applicable. Then, Eq. (46) becomes

$$\frac{d^4W}{dR^4}+\frac{2}{R}\frac{d^3W}{dR^3}-\frac{1}{R^2}\frac{d^2W}{dR^2}+\frac{1}{R^3}\frac{dW}{dR}-\bar{\omega}^2W=0 \quad (47)$$

where $R=r/a$ and $\bar{\omega}^2=\rho h a^4\omega^2/D$.

Applying differential quadrature, one obtains

$$\sum_{j=1}^{N} D_{ij}W_j+\frac{2}{R_i}\sum_{j=1}^{N} C_{ij}W_j-\frac{1}{R_i^2}\sum_{j=1}^{N} B_{ij}W_j+\frac{1}{R_i^3}\sum_{j=1}^{N} A_{ij}W_j$$
$$=\bar{\omega}^2W_i \qquad i=2,\ldots,(N-2) \quad (48)$$

with regularity condition

$$\sum_{j=1}^{N} A_{1j}W_j=0 \quad (49)$$

The regularity condition is necessary to assure that the plate slope is zero at the origin. The set of Eqs. (48) and (49) is solved in conjunction with the appropriate boundary conditions at the plate edge to obtain the plate natural frequency.

For a solid circular plate clamped at its edge,

$$W(1)=0, \qquad \frac{dW}{dR}(1)=0 \quad (50)$$

Numerical results obtained by various methods for this case are listed in Table 5.

For rectangular plates, Eq. (43) holds and Eq. (46) becomes

$$\frac{\partial^4W}{\partial X^4}+2\alpha^2\frac{\partial^2W}{\partial X^2\partial Y^2}+\alpha^4\frac{\partial^4W}{\partial Y^4}-\frac{1}{16}\bar{\omega}^2W=0 \quad (51)$$

where

$$\bar{\omega}^2=\rho h(2a)^4\omega^2/D$$

Applying differential quadrature, one obtains

$$\sum_{k=1}^{N} D_{ik}W_{kj}+2\alpha^2\sum_{\ell=1}^{N} B_{j\ell}\sum_{k=1}^{N} B_{ik}W_{k\ell}+\alpha^4\sum_{k=1}^{N} D_{jk}W_{ik}$$
$$=\frac{1}{16}\bar{\omega}^2W_{ij} \qquad i,j=3,4,\ldots,(N-2) \quad (52)$$

This set of equations together with the appropriate boundary conditions can be solved for the fundamental frequency of a rectangular plate. For a plate clamped on all four edges, these conditions are

$$W(X,\pm1)=W(\pm1,Y)=0$$
$$\frac{\partial W}{\partial Y}(X,\pm1)=\frac{\partial W}{\partial X}(\pm1,Y)=0 \quad (53)$$

Numerical results for a clamped square plate are given in Table 6.

Table 5 Summary of numerical results for fundamental frequency of flexural vibration of a solid circular plate clamped at the edge (exact result: $\bar{\omega}$ = 10.216)

Method	$\bar{\omega}$	% Error
Rayleigh ($n=2$)[a]	10.328	1.10
Rayleigh-Schmidt ($n=1.85$)[a]	10.246	0.29
Present: Differential quadrature (8 grid points)	10.197	−0.19

[a]Ref. 27.

Table 6 Summary of numerical results for fundamental frequency of flexural vibration of a square plate clamped on all four edges (exact result: $\bar{\omega}$ = 35.98 +)[a]

Method	$\bar{\omega}$	% Error
Rayleigh-Ritz (4-term polynomial)[b]	36.00	0.056
Rayleigh-Ritz (36-term trig.)[c]	35.99	0.028
Present: Differential quadrature (5 × 5 grid points)	35.782	−0.550

[a]Ref. 28. [b]Ref. 29. [c]Ref. 30.

Interpretation of Results and Concluding Remarks

As can be seen from the results listed in Tables 1–3, the new energy method is at least as accurate as any of the other approximate methods, with the exception of the differential quadrature method in Table 1. Unfortunately, however, the NEM is not amenable to problems involving more than one dimension.

In Tables 1–6, the DQM is at least as accurate as any of the other approximate methods in three of the six cases. All of the DQM results are computed with a grid of no more than eight points in one direction. In problems governed by differential equations involving up to second derivatives with respect to position (Tables 1, 3, and 4), the grid points are chosen to be equally spaced. However, in problems governed by differential equations involving up to fourth derivatives with respect to position (Tables 2, 5, and 6), equal spacing of grid points would lead to appreciable error in the application of one of the two boundary conditions at each edge unless an excessive number of grid points is used. Thus, in these problems, it is advantageous to use nonuniform grid point spacing such that, for each boundary point, there is an interior point in close proximity and lying along the normal to the boundary at the boundary point. In this way, the two respective boundary conditions can be applied on and very close to each edge. Despite the difficulties mentioned, the DQM was found to require much less computational effort than the finite-element and finite-difference methods to achieve comparable accuracy. It is noted that each of the cases considered required a CPU time of only 1 s or less on an IBM 3081K main frame computer.

Finally, differential quadrature can easily be applied to linear static deflection and buckling problems,[31] as well as to nonlinear structural problems.[32,33]

In conclusion, the two techniques presented here show good promise for further development as practical computational methods in the intermediate range between Rayleigh/Galerkin techniques on one hand and numerical methods such as FEM and BEM on the other.

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